A Direct Transformation Method and its Application to Variable Coefficient Nonlinear Equations of Schrödinger Type

Li-Hua Zhang[a,b] and Xi-Qiang Liu[c]

[a] Department of Mathematics, Shandong University, Jinan 250100, China
[b] Department of Mathematics, Dezhou University, Dezhou 253023, China
[c] School of Mathematical Sciences, Liaocheng University, Liaocheng 252059, China

Reprint requests to Dr. L.-H. Z.; Fax: +86 534 8985615;
E-mail: zzlh100@163.com or zhanglihuazlh100@126.com

Z. Naturforsch. **64a,** 697 – 708 (2009); received September 12, 2008 / revised December 10, 2008

In this paper, the generalized variable coefficient nonlinear Schrödinger (NLS) equation and the cubic-quintic nonlinear Schrödinger (CQNLS) equation with variable coefficients are directly reduced to simple and solvable ordinary differential equations by means of a direct transformation method. Taking advantage of the known solutions of the obtained ordinary differential equations, families of exact nontravelling wave solutions for the two equations have been constructed. The characteristic feature of the direct transformation method is, that without much extra effort, we circumvent the integration by directly reducing the variable coefficient nonlinear evolution equations to the known ordinary differential equations. Another advantage of the method is that it is independent of the integrability of the given nonlinear equation. The method used here can be applied to reduce other variable coefficient nonlinear evolution equations to ordinary differential equations.

Key words: Direct Transformation Method; Generalized Variable Coefficient Nonlinear Schrödinger (NLS) Equation; Cubic-quintic Nonlinear Schrödinger (CQNLS) Equation with Variable Coefficients; Nontravelling Wave Solutions.

1. Introduction

In the past decades, the nonlinear Schrödinger (NLS) equation and its generalized forms have been extensively studied [1 – 18]. Nonlinear equations of Schrödinger type are of great interest due to their central importance to the theory of quantum mechanics [7]. They arise in plasma waves and nonlinear optics, and are of importance in the development of soliton and inverse scattering transform theory [8, 19]. As an important model in optical communication systems, the nonlinear Schrödinger model with variable coefficients has directed the attention of many researchers. Tian and Gao [11] applied a direct method to get exact, analytic bright-solitonic solutions for the perturbed nonlinear Schrödinger model. In [12 – 13] the variable coefficient higher order nonlinear Schrödinger equation has been studied from the viewpoint of bilinear form, Backlund transformation, brightons and symbolic computation. The authors in [14] discussed generation, compression and propagation of pulse trains in the nonlinear Schrödinger equation with distributed coefficients.

On the other hand, directly seeking for exact solutions of nonlinear partial differential equations (NLP DEs) has become one of central theme of perpetual interest in mathematical physics and many effective methods have been presented such as inverse scattering method [8, 19], Hirota's bilinear method [20], Backlund transformation method [21], Painleve expansion method [22], and so on. Recently, based on the computer system like Maple or Mathematica, many powerful algebraic methods, which include sine-cosine method [23], the Riccati equation expansion method [24], the Jacobi elliptic function expansion method [25], exp method [26], and so on [27 – 31], have been proposed and applied to derive many exact solutions of NLPDEs. Generally speaking, exact solutions of NLPDEs obtained by those algebraic methods are written as a polynomial of special solutions which satisfy simple and solvable nonlinear ordinary differential equations (ODEs) such as Riccati equation, the first kind and the second kind elliptic equation, the first-order nonlinear ODE with six degree nonlinear terms, etc. Motivated by the idea of those algebraic methods, in this paper, we aim to reduce the variable coefficient nonlinear equations of Schrödinger type to ordinary differential equations by means of a direct transformation method.

The paper is organized as follows: In Section 2, the generalized variable coefficient nonlinear Schrödinger (NLS) equation and in Section 3 the cubic-quintic nonlinear Schrödinger (CQNLS) equation with variable coefficients are chosen to illustrate our method and families of explicit nontravelling wave solutions are obtained which include bright and dark soliton solutions, triangular periodic solutions, Jacobi elliptic function solutions and rational function solutions. Some conclusions and discussions are given in Section 4.

2. Nontravelling Wave Solutions of the Generalized Variable Coefficient NLS Equation

In this section, we study the exact nontravelling wave solutions of the following generalized variable coefficient NLS equation [1–5, 14–18]:

$$\mathrm{i}u_x + b(x)u_{tt} + l(x)|u|^2u + \mathrm{i}d(x)u = 0, \tag{1}$$

where $u(x,t)$ is the complex envelope of the electrical field in a reference frame moving with the pulse, x is the normalized propagation distance and z is the retarded time. $b(x)$ represents the group velocity dispersion, $l(x)$ is the nonlinearity parameter, and $d(x)$ is the amplification or absorption coefficient. These are all real functions of x. In practical applications, (1) can be used to describe not only the amplification or absorption of optical pulses propagating in inhomogeneous optical fiber systems, but also the stable transmission of managed solitons. Exact chirped gray soliton solutions of (1) have been found in [1] by using a transformation of variables and functions. Soliton solutions and the Backlund transformation for (1) have been studied in [2] and multisoliton solutions in terms of double Wronskian determinant for (1) have been derived in [3]. Families of exact solutions of (1) have been obtained in [4] by the variable coefficient F-expansion method. Generation, compression and propagation of pulse trains of (1) have been discussed in [14]. By constructing four transformations, relations between (1) and the well-known standard NLS equation and cylindrical NLS equation have been built in [15]. For more details about the results of (1), the readers are advised to see [16–18]. Here we try to deal with (1) by the direct transformation method and give a series of new exact nontravelling wave solutions for (1).

In order to change (1) into an ordinary differential equation, we make the first transformation

$$u(x,t) = v(x,t)\mathrm{e}^{\mathrm{i}w(x,t)}, \tag{2}$$

where $v(x,t)$ and $w(x,t)$ are real functions. Substituting (2) into (1) and setting the real part and imaginary part to zero, respectively, we can get

$$v_x + 2b(x)w_t v_t + (b(x)w_{tt} + d(x))v = 0, \tag{3a}$$

$$-vw_x + b(x)v_{tt} - b(x)vw_t^2 + l(x)v^3 = 0. \tag{3b}$$

Note that (3a) is a linear equation with respect to v and its derivatives. To get v from (3a), we make the second transformation

$$w(x,t) = h_1(x)t^2 + h_2(x)t + h_3(x), \tag{4}$$

where $h_1(x)$, $h_2(x)$, and $h_3(x)$ are undetermined functions. It follows from (3a) that

$$v_x + (2bh_2 + 4bh_1t)v_t + (2bh_1 + d)v = 0. \tag{5}$$

Solving (5) by its corresponding characteristic equation, we get

$$v = U(\phi)\exp\left[-\int(2bh_1 + d)\mathrm{d}x\right], \tag{6}$$

where $\phi = t\exp[-\int 4bh_1\mathrm{d}x] - \exp\{2bh_2\exp[-\int 4bh_1\mathrm{d}x]\}$, $U(\phi)$ is an arbitrary function. Substituting (6) into (3b) and reducing, it follows from (3b) that

$$\begin{aligned} U'' = {} & \frac{-l(x)\exp[\int 4bh_1\mathrm{d}x]}{b\exp[\int 2d(x)\mathrm{d}x]}U^3 \\ & + \frac{\exp[\int 8bh_1\mathrm{d}x]}{b}\Big[(h_{1x} + 4bh_1{}^2)t^2 \\ & + (h_{2x} + 4bh_1h_2)t + (h_{3x} + bh_2{}^2)\Big]U. \end{aligned} \tag{7}$$

Thus, looking for exact solutions of (3) leads to exact solutions of (7). To reduce (7) to an ordinary differential equation, let us take the following constraint conditions:

$$\frac{-l(x)\mathrm{e}^{\int 4bh_1\mathrm{d}x}}{b\mathrm{e}^{\int 2d(x)\mathrm{d}x}} = 2a_4, \tag{8a}$$

$$\begin{aligned} & \frac{\exp[\int 8bh_1\mathrm{d}x]}{b}(h_{1x} + 4bh_1{}^2)t^2 + (h_{2x} + 4bh_1h_2)t \\ & + (h_{3x} + bh_2{}^2) = a_2, \end{aligned} \tag{8b}$$

where a_2 and a_4 are arbitrary constants. Under condition (8), it follows from (7) that

$$U'^2 = a_0 + a_2 U^2 + a_4 U^4, \tag{9}$$

where a_0 is an arbitrary constant.

From (8a), we have the following constraint condition imposed on the coefficient functions of (1):

$$l(x) = \frac{-2a_4 b \mathrm{e}^{\int 2d\mathrm{d}x}}{\int 4b\mathrm{d}x + G_1}. \tag{10a}$$

It is straightforward to solve (8b). The result reads

$$h_1 = \frac{1}{\int 4b\mathrm{d}x + G_1}, \quad h_2 = \frac{G_2}{\int 4b\mathrm{d}x + G_1}, \quad h_3 = \frac{\frac{1}{4}(G_2{}^2 - a_2)}{\int 4b\mathrm{d}x + G_1}, \tag{10b}$$

where G_1 and G_2 are arbitrary constants, b and d are the arbitrary coefficient functions, $\int 4b\mathrm{d}x + G_1 > 0$. With the help of (10a) and (10b), it follows from (4) and (6) that

$$w = \frac{1}{4}\frac{4t^2 + 4G_2 t + G_2{}^2 - a_2}{\int 4b\mathrm{d}x + G_1}, \quad v = U(\phi)\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x\right], \tag{11a}$$

$$\phi = \frac{t + \frac{1}{2}G_2}{\int 4b\mathrm{d}x + G_1}. \tag{11b}$$

In summary, the following theorem holds:

Theorem 1. If the coefficient functions b, l, and d in (1) satisfy relation (10a), then solutions of (1) can be expressed by

$$u = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2 t + G_2{}^2 - a_2}{\int 4b\mathrm{d}x + G_1}\right]U(\phi), \tag{12}$$

where $U(\phi)$ is a solution of (9) and ϕ is determined by (11b).

Taking advantage of Theorem 1 and the known solutions of (9) [31], we can give exact nontravelling wave solutions of (1) as follows:

Case A. Jacobi elliptic function solutions and combined Jacobi elliptic function solutions

(A.1) $a_0 = 1, \quad a_2 = -(k^2+1), \quad a_4 = k^2,$

$$u_{\mathrm{A.1.1}} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2 t + G_2{}^2 + 1 + k^2}{\int 4b\mathrm{d}x + G_1}\right]\mathrm{sn}(\phi),$$

$$u_{\mathrm{A.1.2}} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2 t + G_2{}^2 + 1 + k^2}{\int 4b\mathrm{d}x + G_1}\right]\mathrm{cd}(\phi),$$

(A.2) $a_0 = k^2, \quad a_2 = -1 - k^2, \quad a_4 = 1,$

$$u_{\mathrm{A.2.1}} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2 t + G_2{}^2 + 1 + k^2}{\int 4b\mathrm{d}x + G_1}\right]\mathrm{ns}(\phi),$$

$$u_{\mathrm{A.2.2}} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2 t + G_2{}^2 + 1 + k^2}{\int 4b\mathrm{d}x + G_1}\right]\mathrm{dc}(\phi),$$

(A.3) $a_0 = k^2 - 1, \quad a_2 = 2 - k^2, \quad a_4 = -1,$

$$u_{\mathrm{A.3}} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2 t + G_2{}^2 - (2 - k^2)}{\int 4b\mathrm{d}x + G_1}\right]\mathrm{dn}(\phi),$$

(A.4) $a_0 = 1 - k^2, \quad a_2 = 2k^2 - 1, \quad a_4 = -k^2,$

$$u_{\mathrm{A.4}} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2 t + G_2{}^2 - (2k^2 - 1)}{\int 4b\mathrm{d}x + G_1}\right]\mathrm{cn}(\phi),$$

(A.5) $a_0 = -k^2, \quad a_2 = -1 + 2k^2, \quad a_4 = 1 - k^2,$

$$u_{\mathrm{A.5}} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2 t + G_2{}^2 - (-1 + 2k^2)}{\int 4b\mathrm{d}x + G_1}\right]\mathrm{nc}(\phi),$$

(A.6) $a_0 = -1, \quad a_2 = 2 - k^2, \quad a_4 = k^2 - 1,$

$$u_{\mathrm{A.6}} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2 t + G_2{}^2 - (2 - k^2)}{\int 4b\mathrm{d}x + G_1}\right]\mathrm{nd}(\phi),$$

(A.7) $a_0 = 1-k^2, \quad a_2 = 2-k^2, \quad a_4 = 1,$

$$u_{A.7} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-(2-k^2)}{\int 4b\mathrm{d}x+G_1}\right]\mathrm{cs}(\phi),$$

(A.8) $a_0 = 1, \quad a_2 = 2-k^2, \quad a_4 = 1-k^2,$

$$u_{A.8} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-(2-k^2)}{\int 4b\mathrm{d}x+G_1}\right]\mathrm{sc}(\phi),$$

(A.9) $a_0 = 1, \quad a_2 = 2k^2-1, \quad a_4 = k^2(k^2-1),$

$$u_{A.9} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-(2k^2-1)}{\int 4b\mathrm{d}x+G_1}\right]\mathrm{sd}(\phi),$$

(A.10) $a_0 = k^2(k^2-1), \quad a_2 = 2k^2-1, \quad a_4 = 1,$

$$u_{A.10} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-(2k^2-1)}{\int 4b\mathrm{d}x+G_1}\right]\mathrm{ds}(\phi),$$

(A.11) $a_0 = \frac{1}{4}, \quad a_2 = \frac{1-2k^2}{2}, \quad a_4 = \frac{1}{4},$

$$u_{A.11} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{1-2k^2}{2}}{\int 4b\mathrm{d}x+G_1}\right][\mathrm{ns}(\phi)\pm\mathrm{cs}(\phi)],$$

(A.12) $a_0 = \frac{1-k^2}{4}, \quad a_2 = \frac{1+k^2}{2}, \quad a_4 = \frac{1-k^2}{4},$

$$u_{A.12} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{1+k^2}{2}}{\int 4b\mathrm{d}x+G_1}\right][\mathrm{nc}(\phi)\pm\mathrm{sc}(\phi)],$$

(A.13) $a_0 = \frac{k^4}{4}, \quad a_2 = \frac{k^2-2}{2}, \quad a_4 = \frac{1}{4},$

$$u_{A.13} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{k^2-2}{2}}{\int 4b\mathrm{d}x+G_1}\right][\mathrm{ns}(\phi)+\mathrm{ds}(\phi)],$$

(A.14) $a_0 = \frac{k^2}{4}, \quad a_2 = \frac{k^2-2}{2}, \quad a_4 = \frac{k^2}{4},$

$$u_{A.14.1} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{k^2-2}{2}}{\int 4b\mathrm{d}x+G_1}\right][\mathrm{sn}(\phi)\pm\mathrm{i}\,\mathrm{cn}(\phi)],$$

$$u_{A.14.2} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{k^2-2}{2}}{\int 4b\mathrm{d}x+G_1}\right]\frac{\mathrm{dn}(\phi)}{\mathrm{i}\sqrt{1-k^2}\mathrm{sn}(\phi)\pm\mathrm{cn}(\phi)},$$

(A.15) $a_0 = 1, \quad a_2 = 2-4k^2, \quad a_4 = 1,$

$$u_{A.15} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-(2-4k^2)}{\int 4b\mathrm{d}x+G_1}\right]\frac{\mathrm{sn}(\phi)\mathrm{dn}(\phi)}{\mathrm{cn}(\phi)},$$

(A.16) $a_0 = \frac{k^2-2k+1}{4A^2}, \quad a_2 = \frac{1+k^2}{2}+3k, \quad a_4 = \frac{A^2(-1+k)^2}{4},$

$$u_{A.16} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\left(\frac{1+k^2}{2}+3k\right)}{\int 4b\mathrm{d}x+G_1}\right] \cdot \frac{\mathrm{dn}(\phi)\mathrm{cn}(\phi)}{A(1+\mathrm{sn}(\phi))(1+k\mathrm{sn}(\phi))},$$

(A.17) $a_0 = \frac{(k+1)^2}{4A^2}, \quad a_2 = \frac{1+k^2}{2}-3k, \quad a_4 = \frac{A^2(1+k)^2}{4},$

$$u_{A.17} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\left(\frac{1+k^2}{2}-3k\right)}{\int 4b\mathrm{d}x+G_1}\right] \cdot \frac{\mathrm{dn}(\phi)\mathrm{cn}(\phi)}{A(1+\mathrm{sn}(\phi))(1-k\mathrm{sn}(\phi))},$$

(A.18) $a_0 = -2k^3 + k^4 + k^2, \quad a_2 = 6k - k^2 - 1, \quad a_4 = -\frac{4}{k},$

$$u_{A.18} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2{}^2 - (6k - k^2 - 1)}{\int 4b\mathrm{d}x + G_1}\right]\frac{k\mathrm{dn}(\phi)\mathrm{cn}(\phi)}{1 + k\mathrm{sn}^2(\phi)},$$

(A.19) $a_0 = 2k^3 + k^4 + k^2, \quad a_2 = -6k - k^2 - 1, \quad a_4 = \frac{4}{k},$

$$u_{A.19} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2{}^2 + 6k + k^2 + 1}{\int 4b\mathrm{d}x + G_1}\right]\frac{k\mathrm{dn}(\phi)\mathrm{cn}(\phi)}{k\mathrm{sn}^2(\phi) - 1},$$

(A.20) $a_0 = 2 + 2k_1 - k^2, \quad a_2 = 6k_1 - k^2 + 2, \quad a_4 = 4k_1,$

$$u_{A.20} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2{}^2 - (6k_1 - k^2 + 2)}{\int 4b\mathrm{d}x + G_1}\right]\frac{k^2\mathrm{sn}(\phi)\mathrm{cn}(\phi)}{k_1 - \mathrm{dn}^2(\phi)},$$

(A.21) $a_0 = 2 - 2k_1 - k^2, \quad a_2 = -6k_1 - k^2 + 2, \quad a_4 = -4k_1,$

$$u_{A.21} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2{}^2 - (-6k_1 - k^2 + 2)}{\int 4b\mathrm{d}x + G_1}\right]\frac{-k^2\mathrm{sn}(\phi)\mathrm{cn}(\phi)}{k_1 + \mathrm{dn}^2(\phi)},$$

(A.22) $a_0 = \frac{k^2 - 1}{4(C^2k^2 - B^2)}, \quad a_2 = \frac{k^2 + 1}{2}, \quad a_4 = \frac{(C^2k^2 - B^2)(k^2 - 1)}{4},$

$$u_{A.22} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2{}^2 - \frac{k^2+1}{2}}{\int 4b\mathrm{d}x + G_1}\right]\frac{\sqrt{\frac{B^2 - C^2}{B^2 - C^2k^2}} + \mathrm{sn}(\phi)}{B\mathrm{cn}(\phi) + C\mathrm{dn}(\phi)},$$

(A.23) $a_0 = \frac{k^4}{4(C^2 + B^2)}, \quad a_2 = \frac{k^2}{2} - 1, \quad a_4 = \frac{C^2 + B^2}{4},$

$$u_{A.23} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2{}^2 - \left(\frac{k^2}{2} - 1\right)}{\int 4b\mathrm{d}x + G_1}\right]\frac{\sqrt{\frac{B^2 + C^2 - C^2k^2}{B^2 + C^2}} + \mathrm{dn}(\phi)}{B\mathrm{sn}(\phi) + C\mathrm{cn}(\phi)},$$

(A.24) $a_0 = \frac{2k - k^2 - 1}{B^2}, \quad a_2 = 2k^2 + 2, \quad a_4 = -B^2k^2 - B^2 - 2B^2k,$

$$u_{A.24} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2{}^2 - (2k^2 + 2)}{\int 4b\mathrm{d}x + G_1}\right]\frac{k\mathrm{sn}^2(\phi) - 1}{B(k\mathrm{sn}^2(\phi) + 1)},$$

(A.25) $a_0 = -\frac{2k + k^2 + 1}{B^2}, \quad a_2 = 2k^2 + 2, \quad a_4 = -B^2k^2 - B^2 + 2B^2k,$

$$u_{A.25} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2{}^2 - (2k^2 + 2)}{\int 4b\mathrm{d}x + G_1}\right]\frac{k\mathrm{sn}^2(\phi) + 1}{B(k\mathrm{sn}^2(\phi) - 1)},$$

(A.26) $a_0 = a_4 = \frac{1}{4}, \quad a_2 = \frac{1 - 2k^2}{2},$

$$u_{A.26.1} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2{}^2 - \frac{1-2k^2}{2}}{\int 4b\mathrm{d}x + G_1}\right]\mathrm{dn}(\phi)\left[k\mathrm{cn}(\phi) \pm \mathrm{i}\sqrt{1 - k^2}\right],$$

$$u_{A.26.2}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{1-2k^2}{2}}{\int 4b\mathrm{d}x+G_1}\right][k\mathrm{sn}(\phi)\pm\mathrm{i}\mathrm{dn}(\phi)],$$

$$u_{A.26.3}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{1-2k^2}{2}}{\int 4b\mathrm{d}x+G_1}\right]\frac{\mathrm{sn}(\phi)}{1\pm\mathrm{cn}(\phi)},$$

$$u_{A.26.4}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{1-2k^2}{2}}{\int 4b\mathrm{d}x+G_1}\right]\frac{\mathrm{cn}(\phi)}{\sqrt{1-k^2}\mathrm{sn}(\phi)\pm\mathrm{dn}(\phi)},$$

(A.27) $a_0=a_4=\dfrac{k^2-1}{4},\quad a_2=\dfrac{k^2+1}{2},$

$$u_{A.27.1}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{k^2+1}{2}}{\int 4b\mathrm{d}x+G_1}\right]\frac{\mathrm{dn}(\phi)}{1\pm k\mathrm{sn}(\phi)},$$

$$u_{A.27.2}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{k^2+1}{2}}{\int 4b\mathrm{d}x+G_1}\right][k\mathrm{sd}(\phi)\pm\mathrm{nd}(\phi)],$$

(A.28) $a_0=a_4=\dfrac{1-k^2}{4},\quad a_2=\dfrac{k^2+1}{2},$

$$u_{A.28.1}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{k^2+1}{2}}{\int 4b\mathrm{d}x+G_1}\right]\frac{\mathrm{cn}(\phi)}{1\pm\mathrm{sn}(\phi)},$$

$$u_{A.28.2}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{k^2+1}{2}}{\int 4b\mathrm{d}x+G_1}\right][\mathrm{nc}(\phi)\pm\mathrm{sc}(\phi)],$$

(A.29) $a_0=-\dfrac{(1-k^2)^2}{4},\quad a_2=\dfrac{k^2+1}{2},\quad a_4=\dfrac{-1}{4},$

$$u_{A.29}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{k^2+1}{2}}{\int 4b\mathrm{d}x+G_1}\right][k\mathrm{cn}(\phi)\pm\mathrm{dn}(\phi)],$$

(A.30) $a_0=\dfrac{1}{4},\quad a_2=\dfrac{k^2+1}{2},\quad a_4=\dfrac{(1-k^2)^2}{4},$

$$u_{A.30}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{k^2+1}{2}}{\int 4b\mathrm{d}x+G_1}\right]\frac{\mathrm{sn}(\phi)}{\mathrm{dn}(\phi)\pm\mathrm{cn}(\phi)},$$

(A.31) $a_0=\dfrac{1}{4},\quad a_2=\dfrac{k^2-2}{2},\quad a_4=\dfrac{k^4}{4},$

$$u_{A.31}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-\frac{k^2-2}{2}}{\int 4b\mathrm{d}x+G_1}\right]\frac{\mathrm{cn}(\phi)}{\sqrt{1-k^2}\pm\mathrm{dn}(\phi)}.$$

Case B. Soliton solutions and combined soliton solutions

(B.1) When $a_0=0$, $a_2>0$, $a_4<0$, (1) has the following bright soliton solution:

$$u_{B.1}=\exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x+G_1}+d\right)\mathrm{d}x+\frac{\mathrm{i}}{4}\frac{4t^2+4G_2t+G_2{}^2-a_2}{\int 4b\mathrm{d}x+G_1}\right]\sqrt{\frac{-a_2}{a_4}}\mathrm{sech}\left(\sqrt{a_2}\phi\right).$$

(B.2) When $a_0 = 0$, $a_2 > 0$, $a_4 > 0$, (1) has the following singular solution:

$$u_{B.2} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2^2 - a_2}{\int 4b\mathrm{d}x + G_1}\right]\sqrt{\frac{a_2}{a_4}}\operatorname{csch}\left(\sqrt{a_2}\phi\right).$$

(B.3) When $a_0 = \dfrac{a_2^2}{4a_4}$, $a_2 < 0$, $a_4 > 0$, (1) has the following dark soliton solution:

$$u_{B.3} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2^2 - a_2}{\int 4b\mathrm{d}x + G_1}\right]\sqrt{\frac{-a_2}{2a_4}}\tanh\left(\sqrt{-\frac{a_2}{2}}\phi\right).$$

(B.4) When $a_0 = 0$, $a_2 = 1$, $a_4 = \dfrac{1}{2}$, (1) has the following dark soliton solution:

$$u_{B.4} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2^2 - 1}{\int 4b\mathrm{d}x + G_1}\right]\frac{\pm\sqrt{2 - 2\tanh^2(G-\phi)}}{\tanh(G-\phi)}.$$

(B.5) When $k \to 1$, the Jacobi elliptic function solution $u_{A.1.1}$ degenerates as dark soliton solution:

$$u_{B.5} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2^2 + 2}{\int 4b\mathrm{d}x + G_1}\right]\tanh(\phi).$$

(B.6) When $k \to 1$ the Jacobi elliptic function solution $u_{A.4}$ degenerates as bright soliton solution:

$$u_{B.6} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2^2 - 1}{\int 4b\mathrm{d}x + G_1}\right]\operatorname{sech}(\phi).$$

Case C. Triangular periodic solutions

(C.1) When $a_0 = 0$, $a_2 < 0$, $a_4 > 0$, (1) has the following triangular periodic solutions:

$$u_{C.11} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2^2 - a_2}{\int 4b\mathrm{d}x + G_1}\right]\sqrt{\frac{-a_2}{a_4}}\sec\left(\sqrt{-a_2}\phi\right).$$

$$u_{C.12} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2^2 - a_2}{\int 4b\mathrm{d}x + G_1}\right]\sqrt{\frac{-a_2}{a_4}}\csc\left(\sqrt{-a_2}\phi\right).$$

(C.2) When $a_0 = \dfrac{a_2^2}{4a_4}$, $a_2 > 0$, $a_4 > 0$, (1) has the following triangular periodic solutions:

$$u_{C.2} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2^2 - a_2}{\int 4b\mathrm{d}x + G_1}\right]\sqrt{\frac{a_2}{2a_4}}\tan\left(\sqrt{\frac{a_2}{2}}\xi\right).$$

(C.3) When $k \to 0$ the Jacobi elliptic function solution $u_{A.8}$ degenerates as triangular periodic solution:

$$u_{C.3} = \exp\left[-\int\left(\frac{2b}{\int 4b\mathrm{d}x + G_1} + d\right)\mathrm{d}x + \frac{\mathrm{i}}{4}\frac{4t^2 + 4G_2t + G_2^2 - 2}{\int 4b\mathrm{d}x + G_1}\right]\tan(\phi),$$

where k $(0 < k < 1)$ denotes the modulus of the Jacobi elliptic function, $k_1 = \sqrt{1-k^2}$, $\mathrm{i}^2 = -1$, G_1, G_2, A, B, C $(ABC \neq 0)$, and G are arbitrary constants, the coefficient functions b and d are arbitrary functions of x, $l(x)$ is determined by (10a).

As aforementioned, in a realistic optical fibre, when the characteristic parameters of the fibre are not constants, the variations of the group velocity dispersion, the nonlinearity, and the amplification or absorption can be described by (1). Considering the dark soliton solution $u_{B.5}$ of (1), we can see that it describes the chirped dark soliton pulse for (1), where $\tanh\left(\frac{t+\frac{1}{2}G_2}{\int 4b\mathrm{d}x+G_1}\right)$ represents the shape of the dark soliton, $\frac{-G_2}{2}$ is the center of the soliton, $\frac{1}{\int 4b\mathrm{d}x+G_1}$ are the inverse width of the soliton and the chirp parameter. Similarly, we observe that the bright soliton solution $u_{B.6}$ describes the chirped bright soliton pulse for (1), where $\operatorname{sech}\left(\frac{t+\frac{1}{2}G_2}{\int 4b\mathrm{d}x+G_1}\right)$ represents the shape of the bright soliton, $\frac{-G_2}{2}$ is the center of the pulse, and $\frac{1}{\int 4b\mathrm{d}x+G_1}$ are the inverse width of the soliton and the chirp parameter. From the chirped dark and bright soliton solution $u_{B.5}$ and $u_{B.6}$, we see that the chirped term $\frac{1}{\int 4b\mathrm{d}x+G_1}$ and the amplification or absorption coefficient $d(x)$ determine the qualities of the solitary wave.

Remark 1. When $k \to 1$, the Jacobi elliptic function degenerates as hyperbolic function in the manner of Appendix 1. From the Jacobi elliptic function solutions obtained in Case A, we can get the other soliton solutions for (1) in addition to $u_{B.5}$ and $u_{B.6}$. When $k \to 0$, the Jacobi elliptic function degenerates as triangular periodic function in the manner of Appendix 1. From the Jacobi elliptic function solutions obtained in Case A, we can get the other triangular periodic solutions for (1) in addition to $u_{C.2}$. For simplicity, we omit them here.

Remark 2. If the normalized propagation distance z in [4] becomes x and $A_1 = G_2$, $A_2 = 1$, $A_3 = 1$, $A_4 = A_5 = 0$, $\alpha(x) = 2b(x)$, $\gamma(x) = -d(x)$, $\kappa = \frac{1}{\int 4b(x)\mathrm{d}x+G_1}$, $c_0 = a_0$, $c_2 = a_2$, and $c_4 = \frac{a_4}{2}$, then the solutions ψ_{13}, ψ'_{13}, ψ_{22}, ψ'_{22}, ψ_{33}, and ψ'_{33} in [4] are exactly the same as those obtained in this paper, named $u_{A.1.1}$, $u_{B.5}$, $u_{A.4}$, $u_{B.6}$, $u_{A.8}$, and $u_{C.3}$, respectively. In fact, the exact solutions $u_{A.1.1}$ to $u_{A.10}$ of (1), which are expressed by the single Jacobi elliptic function, can all be obtained in [4] with the above condition and in [16] when the variable x becomes z and $b(z) = \frac{\beta(z)}{2}$, $l(z) = \gamma(z)$, $d(z) = -\alpha(z)$, $G_1 = 0$, $C_0 = \frac{1}{2}$, $C_1 = \frac{G_2}{4}$, $\lambda = a_2$, and $\mu = -4a_4$. Comparing our work with the results in [4] and [16], we present more Jacobi doubly periodic solutions and soliton solutions and our computation is much simpler.

Remark 3. If the variable x becomes z and $b(z) = \pm\frac{D(z)}{2}$, $l(z) = R(z)$, $d(z) = \Gamma(z)$, $G_1 = \frac{2}{C_0}$, the constraint condition (10a) is the same as (4) in [1] and (4) in [14], respectively. As stated in [1] and [14], (10a) is the condition for (1) to be transformed to the standard NLS equation. It shows that (1) can be reduced to ordinary differential equation (9) when it can be transformed to the standard NLS equation.

3. Nontravelling Wave Solutions of the CQNLS Equation with Variable Coefficients

In this section, we investigate the following cubic-quintic nonlinear Schrödinger (CQNLS) equation with variable coefficients [9–10]

$$\mathrm{i}u_x + \frac{D(x)}{2}u_{tt} + R(x)|u|^2u - \alpha(x)|u|^4u = 0, \quad (13)$$

where $u(x,t)$ is the complex envelope of the electrical field in a comoving frame, $D(x)$ represents the group velocity dispersion, $R(x)$ is the Kerr nonlinearity parameter and the parameter $\alpha(x)$ is the saturation of the nonlinear refractive index (i. e. higher order nonlinearity). They are real functions of the normalized propagation distance x. And t is the retarded time. The CQNLS equation (13) describes the amplification or absorption of pulses propagating in high optical intensities or materials with high nonlinear coefficients (e. g. semiconductor doped glasses) with distributed parameters. In particular, when $\alpha(x) = 0$, (13) can be reduced to the NLS equation with variable coefficients. And when $D(x) = 1$, $R(x)$ and $\alpha(x)$ are constants, (13) becomes the constant coefficient CQNLS equation, whose solutions have been found in [9]. Exact solutions of the special case for (1) when $R(x) = \frac{c_1}{2}D(x)$ and $\alpha(x) = \frac{c_2}{2}D(x)$ have been studied in [10]. Here our proposed method will give families of nontravelling wave solutions for (13) in a unified way.

For our purpose, we first introduce the following transformation:

$$u(x,t) = p(x,t)\exp[\mathrm{i}q(x,t)], \quad (14)$$

where $p(x,t)$ and $q(x,t)$ are real functions. Substituting (14) into (13) and letting the real part and imaginary part be zero, respectively, we can get

$$p_x + D(x)p_tq_t + \frac{D(x)}{2}pq_{tt} = 0, \quad (15a)$$

$$-pq_x+\frac{D(x)}{2}p_{tt}-\frac{D(x)}{2}pq_t{}^2+R(x)p^3-\alpha(x)p^5=0. \tag{15b}$$

Note that (15a) is a linear equation with respect to p and its derivatives. To get p from (15a), we choose

$$q(x,t)=q_1(x)t^2+q_2(x)t+q_3(x), \tag{16}$$

where $q_1(x)$, $q_2(x)$ and $q_3(x)$ are undetermined functions of x. Substituting (16) into (15a), it follows that

$$p=W(\theta)\exp\left[\int -Dq_1\mathrm{d}x\right], \tag{17}$$

where the argument $\theta = t\exp[-\int 2Dq_1\mathrm{d}x] - \int Dq_2\exp[-\int 2Dq_1\mathrm{d}x]\mathrm{d}x$ and $W(\theta)$ is an arbitrary function. Substituting (17) into (15b) and reducing, (15b) is simplified to

$$\begin{aligned} W'' = {} & \frac{2\exp[\int 4Dq_1\mathrm{d}x]}{D}\Big[(q_{1x}+2Dq_1{}^2)t^2 \\ & + (q_{2x}+2Dq_1q_2)t+\left(q_{3x}+\frac{D}{2}q_2{}^2\right)\Big]W \\ & -\frac{2R}{D}\exp\left[\int 2Dq_1\mathrm{d}x\right]W^3+\frac{2\alpha}{D}W^5. \end{aligned} \tag{18}$$

Up to now, seeking for solutions of (13) has been transformed to finding solutions of (18). In order to reduce (18) to an ordinary differential equation, let us take the following constraint conditions:

$$\frac{2\exp[\int 4Dq_1\mathrm{d}x]}{D}\Big[(q_{1x}+2Dq_1{}^2)t^2+(q_{2x}+2Dq_1q_2)t + \left(q_{3x}+\frac{D}{2}q_2{}^2\right)\Big]=c_2, \tag{19a}$$

$$-\frac{2R}{D}\exp\left[\int 2Dq_1\mathrm{d}x\right]=2c_4, \quad \frac{2\alpha}{D}=3c_6, \tag{19b}$$

where c_2, c_4 and c_6 are arbitrary constants. Under the constraint conditions (19a) and (19b), it follows from (18) that

$$W'^2=c_0+c_2W^2+c_4W^4+c_6W^6, \tag{20}$$

where c_0 is an arbitrary constant.

From (19b), the relations between the coefficient functions of (13) are expressed as follows:

$$\alpha=\frac{3D}{2}c_6, \quad R=\frac{-c_4D}{\int 2D\mathrm{d}x+H_2}. \tag{21}$$

It is simple to solve (19a). The result reads:

$$\begin{aligned} & q_1=\frac{1}{\int 2D\mathrm{d}x+H_2}, \quad q_2=\frac{H_1}{\int 2D\mathrm{d}x+H_2}, \\ & q_3=\frac{\frac{1}{4}(H_1{}^2-c_2)}{\int 2D\mathrm{d}x+H_2}+H_3, \end{aligned} \tag{22}$$

where H_1, H_2 and H_3 are arbitrary constants, D is the arbitrary coefficient function, $\int 2D\mathrm{d}x+H_2>0$. Substituting (21) and (22) into (16) and (17), we get

$$\begin{aligned} & p=\frac{W(\theta)}{\sqrt{\int 2D\mathrm{d}x+H_2}}, \quad \theta=\frac{t+\frac{1}{2}H_1}{\int 2D\mathrm{d}x+H_2}, \\ & q=\frac{t^2+H_1t+\frac{1}{4}(H_1{}^2-c_2)}{\int 2D\mathrm{d}x+H_2}+H_3. \end{aligned} \tag{23}$$

In summary, we can arrive at the following theorem.

Theorem 2. If the coefficient functions D, R and α in (13) satisfy relations (21), then solutions of (13) can be expressed by

$$u=\frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x+H_2}}W(\theta), \tag{24}$$

where $W(\theta)$ is a solution of (20), q and θ are determined by (23).

Using Theorem 2 and the known solutions of (20) [31], we can give exact nontravelling wave solutions of (13) as follows.

Case 1. Soliton solutions and combined soliton solutions

(1.1) $c_0=0$, $c_2>0$, $c_4<0$, $c_6<0$, $c_4{}^2-4c_2c_6>0$,

$$u_{1.11}=\frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x+H_2}}\sqrt{\frac{2c_2\operatorname{sech}^2\left(\sqrt{c_2}\theta\right)}{2\sqrt{c_4{}^2-4c_2c_6}-\left(\sqrt{c_4{}^2-4c_2c_6}+c_4\right)\operatorname{sech}^2\left(\sqrt{c_2}\theta\right)}},$$

$$u_{1.12}=\frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x+H_2}}\sqrt{\frac{2c_2\operatorname{csch}^2\left(\pm\sqrt{c_2}\theta\right)}{2\sqrt{c_4{}^2-4c_2c_6}+\left(\sqrt{c_4{}^2-4c_2c_6}-c_4\right)\operatorname{csch}^2\left(\pm\sqrt{c_2}\theta\right)}},$$

(1.2) $c_0 = 0, \quad c_6 = \dfrac{{c_4}^2}{4c_2}, \quad c_2 > 0, \quad c_4 < 0,$

$$u_{1.21} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{-c_2}{c_4}\left(1+\tanh\left(\pm\sqrt{c_2}\theta\right)\right)}, \quad u_{1.22} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{-c_2}{c_4}\left(1+\coth\left(\sqrt{c_2}\theta\right)\right)},$$

(1.3) $c_0 = 0, \quad c_2 > 0,$

$$u_{1.31} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{-c_2c_4\mathrm{sech}^2\left(\sqrt{c_2}\theta\right)}{{c_4}^2 - c_2c_6\left(1\pm\tanh\left(\sqrt{c_2}\theta\right)\right)^2}}, u_{1.32} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{c_2c_4\mathrm{csch}^2\left(\sqrt{c_2}\theta\right)}{{c_4}^2 - c_2c_6\left(1\pm\coth\left(\sqrt{c_2}\theta\right)\right)^2}},$$

(1.4) $c_0 = 0, \quad c_2 > 0, \quad c_6 > 0,$

$$u_{1.41} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{-c_2\mathrm{sech}^2\left(\sqrt{c_2}\theta\right)}{c_4 \pm 2\sqrt{c_2c_6}\tanh\left(\sqrt{c_2}\theta\right)}}, u_{1.42} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{c_2\mathrm{csch}^2\left(\sqrt{c_2}\theta\right)}{c_4 \pm 2\sqrt{c_2c_6}\coth\left(\sqrt{c_2}\theta\right)}},$$

(1.5) $c_0 = \dfrac{8{c_2}^2}{27c_4}, \quad c_6 = \dfrac{{c_4}^2}{4c_2}, \quad c_2 < 0, \quad c_4 > 0,$

$$u_{1.51} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{-8c_2\tanh^2\left(\sqrt{\frac{-c_2}{3}}\theta\right)}{3c_4\left[3+\tanh^2\left(\pm\sqrt{\frac{-c_2}{3}}\theta\right)\right]}}, u_{1.52} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{-8c_2\coth^2\left(\sqrt{\frac{-c_2}{3}}\theta\right)}{3c_4\left[3+\coth^2\left(\pm\sqrt{\frac{-c_2}{3}}\theta\right)\right]}},$$

(1.6) $c_0 = 0, \quad c_2 > 0, \quad {c_4}^2 - 4c_2c_6 > 0,$

$$u_{1.6} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{2c_2}{\pm\sqrt{{c_4}^2 - 4c_2c_6}\cosh\left(2\sqrt{c_2}\theta\right) - c_4}},$$

(1.7) $c_0 = 0, \quad c_2 > 0, \quad {c_4}^2 - 4c_2c_6 < 0,$

$$u_{1.7} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{2c_2}{\pm\sqrt{-{c_4}^2 + 4c_2c_6}\sinh\left(2\sqrt{c_2}\theta\right) - c_4}},$$

Case 2. Triangular periodic solutions

(2.1) $c_0 = 0, \quad c_2 < 0, \quad c_4 \geq 0, \quad c_6 < 0, \quad {c_4}^2 - 4c_2c_6 > 0,$

$$u_{2.11} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{-2c_2\sec^2\left(\sqrt{-c_2}\theta\right)}{2\sqrt{{c_4}^2 - 4c_2c_6} - \left(\sqrt{{c_4}^2 - 4c_2c_6} - c_4\right)\sec^2\left(\sqrt{-c_2}\theta\right)}},$$

$$u_{2.12} = \frac{\exp[iq]}{\sqrt{\int 2D\mathrm{d}x + H_2}}\sqrt{\frac{2c_2\csc^2\left(\pm\sqrt{-c_2}\theta\right)}{2\sqrt{{c_4}^2 - 4c_2c_6} - \left(\sqrt{{c_4}^2 - 4c_2c_6} + c_4\right)\csc^2\left(\pm\sqrt{-c_2}\theta\right)}},$$

(2.2) $c_0 = 0, \quad c_2 < 0, \quad {c_4}^2 - 4c_2c_6 > 0,$

$$u_{2.21} = \frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x + H_2}} \sqrt{\frac{2c_2}{\pm\sqrt{{c_4}^2 - 4c_2c_6}\cos\left(2\sqrt{-c_2}\theta\right) - c_4}},$$

$$u_{2.22} = \frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x + H_2}} \sqrt{\frac{2c_2}{\pm\sqrt{{c_4}^2 - 4c_2c_6}\sin\left(2\sqrt{-c_2}\theta\right) - c_4}},$$

(2.3) $c_0 = 0, \quad c_2 < 0, \quad c_6 > 0,$

$$u_{2.31} = \frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x + H_2}} \sqrt{\frac{-c_2\sec^2\left(\sqrt{-c_2}\theta\right)}{c_4 \pm 2\sqrt{-c_2c_6}\tan\left(\sqrt{-c_2}\theta\right)}}, \quad u_{2.32} = \frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x + H_2}} \sqrt{\frac{-c_2\csc^2\left(\sqrt{-c_2}\theta\right)}{c_4 \pm 2\sqrt{-c_2c_6}\cot\left(\sqrt{-c_2}\theta\right)}},$$

(2.4) $c_0 = \dfrac{8{c_2}^2}{27c_4}, \quad c_6 = \dfrac{{c_4}^2}{4c_4}, \quad c_2 > 0, \quad c_4 < 0,$

$$u_{2.41} = \frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x + H_2}} \sqrt{\frac{8c_2\tan^2\left(\pm\sqrt{\frac{c_2}{3}}\theta\right)}{3c_4\left[3 - \tan^2\left(\pm\sqrt{\frac{c_2}{3}}\theta\right)\right]}}, \quad u_{2.42} = \frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x + H_2}} \sqrt{\frac{8c_2\cot^2\left(\pm\sqrt{\frac{c_2}{3}}\theta\right)}{3c_4\left[3 - \cot^2\left(\pm\sqrt{\frac{c_2}{3}}\theta\right)\right]}},$$

Case 3. Rational function solutions

(3.1) $c_0 = 0, \quad c_2 > 0,$

$$u_{3.1} = \frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x + H_2}} \cdot \sqrt{\frac{16c_2\exp\left[\pm 2\sqrt{c_2}\theta\right]}{\left(\exp\left[\pm 2\sqrt{c_2}\theta\right] - 4c_4\right)^2 - 64c_2c_6}},$$

(3.2) $c_0 = 0, \quad c_2 > 0, \quad c_4 = 0,$

$$u_{3.2} = \frac{\exp[\mathrm{i}q]}{\sqrt{\int 2D\mathrm{d}x + H_2}} \sqrt{\frac{\pm 16c_2\exp\left[\pm 2\sqrt{c_2}\theta\right]}{1 - 64c_2c_6\exp\left[\pm 4\sqrt{c_2}\theta\right]}},$$

where q and θ are determined by (23), the coefficient function D is an arbitrary function of x, α and R are determined by (21).

Remark 4. The results for (13) in this paper are different from that in [10]. To our best knowledge, all solutions for (13) obtained in this paper have not been shown in the existent literature, so they are completely new.

Remark 5. All the solutions obtained in this paper for (1) and (13) have been checked by Maple software.

Auxilary Table. Limits of Jacobi elliptic functions when $k \to 1$ or $k \to 0$.

Jacobi elliptic functions degenerate as hyperbolic functions when $k \to 1$		Jacobi elliptic functions degenerate as hyperbolic functions when $k \to 0$	
sn(ξ)	tanh(ξ)	sn(ξ)	sin(ξ)
cn(ξ)	sech(ξ)	cn(ξ)	cos(ξ)
dn(ξ)	sech(ξ)	dn(ξ)	1
sc(ξ)	sinh(ξ)	sc(ξ)	tan(ξ)
sd(ξ)	sinh(ξ)	sd(ξ)	sin(ξ)
cd(ξ)	1	cd(ξ)	cos(ξ)
ns(ξ)	coth(ξ)	ns(ξ)	csc(ξ)
nc(ξ)	cosh(ξ)	nc(ξ)	sec(ξ)
nd(ξ)	cosh(ξ)	nd(ξ)	1
cs(ξ)	csch(ξ)	cs(ξ)	cot(ξ)
ds(ξ)	csch(ξ)	ds(ξ)	scs(ξ)
dc(ξ)	1	dc(ξ)	sec(ξ)

4. Conclusions and Discussions

In summary, a direct transformation method is developed to find exact solutions of variable coefficient NLPDEs. By means of the proposed method, we have been able to reduce two variable coefficient nonlinear equations of Schrödinger type to two ordinary differential equations under some constraint conditions. Based on the solutions of the ordinary differential equations, we find varieties of exact nontravelling wave solutions for the two equations: the generalized variable coefficient NLS equation (1) and the variable coefficient CQNLS equation (13). These solutions include bright and dark soliton solutions, triangular periodic solutions, Jacobi doubly periodic solutions and rational function solutions. We have also shown that soliton solutions and triangular periodic solutions can be established as the limits of the Jacobi elliptic function solutions. The soliton solutions of (1) and (13), which contain some arbitrary functions and some arbitrary constants, which lead one to discuss the behaviour of solutions and also provide enough freedom to construct solutions that may be related to a real physical problem and can help optical engineers to devise appropriate optical fibre for realizing long distance soliton communication. The transformation technique used in this paper can also be extended to other variable coefficient nonlinear evolution equations.

Acknowledgement

The authors would like to express their sincere thanks to the referees for their many helpful advices and suggestions. This work was supported by the Natural Science Foundation of Shandong Province under Grant No. Y2007G64 and No. Y2008A35.

[1] J. F. Wang, L. Li, and S. T. Jia, Opt. Commun. **274**, 223 (2007).
[2] X. Lü, H. W. Zhu, X. H. Meng, Z. C. Yang, and B. Tian, J. Math. Anal. Appl. **336**, 1305 (2007).
[3] X. Lü, H. W. Zhu, Z. Z. Yao, X. H. Meng, C. Zhang, C. Y. Zhang, and B. Tian, Ann. Phys. **323**, 1947 (2008).
[4] J. F. Zhang, C. Q. Dai, Q. Yang, and J. M. Zhu, Opt. Commun. **252**, 408 (2005).
[5] R. Y. Hao, L. Li, Z. H. Li, W. R. Xue, and G. S. Zhou, Opt. Commun. **236**, 79 (2004).
[6] M. A. Abdou, Chaos, Solitons and Fractals **31**, 95 (2007); C. L. Bai, H. Zhao, and W. T. Wang, Commun. Theor. Phys. **45**, 131 (2006); C. L. Bai, H. Zhao, and L. H. Zhang, Commun, Theor. Phys. **46**, 249 (2006).
[7] M. J. Ablowitz, B. Prinari, and D. A. Trubatch, Nonlinear Schrödinger Equation, Cambridge University Press, Cambridge 2003.
[8] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge 1991.
[9] J. M. Zhu and Z. Y. Ma, Chaos, Solitons and Fractals **33**, 958 (2007).
[10] J. L. Zhang, M. L. Wang, and X. Z. Li, Commun. Theor. Phys. **45**, 343 (2006).
[11] B. Tian and Y. T. Gao, Phys. Lett. A **342**, 228 (2005).
[12] B. Tian and Y. T. Gao, Phys. Lett. A **359**, 241 (2006).
[13] B. Tian, Y. T. Gao, and H. W. Zhu, Phys. Lett. A **366**, 223 (2007).
[14] L. Y. Wang, L. Li, Z. H. Li, and G. S. Zhou, Phys. Rev. E**72**, 036614 (2005).
[15] B. Tian, W. R. Shan, C. Y. Zhang, G. M. Wei, and Y. T. Gao, Eur. Phys. J. B **47**, 329 (2005).
[16] J. L. Zhang, B. A. Li, and M. L. Wang, Chaos, Solitons and Fractals, (In press).
[17] Y. Chen and B. Li, Nonlinear Analysis: Hybrid Syst. **2**, 242 (2008).
[18] V. I. Kruglov, A. C. Peacock, and J. D. Harvey, Phys. Rev. E **71**, 056619 (2005).
[19] M. J. Ablowitz and H. Segur, Soliton and the inverse scattering transformation, SIAM: Philadelphia 1981.
[20] R. Hirota and J. Satsuma, Phys. Lett. A **85**, 407 (1981).
[21] M. Wadati, J. Phys. Soc. Jpn. **38**, 673 (1975).
[22] R. Conte and M. Musette, J. Phys. A: Math. Gen. **22**, 169 (1989).
[23] C. T. Yan, Phys. Lett. A **224**, 77 (1996).
[24] E. G. Fan, Phys. Lett. A **277**, 212 (2000); Z. Y. Yan and H. Q. Zhang, Phys. Lett. A **292**, 100 (2001).
[25] S. K. Liu, Z. T. Fu, S. D. Liu, and Q. Zhao, Phys. Lett. A **289**, 69 (2001); Z. Y. Yan, Chaos, Solitons and Fractals **18**, 299 (2003).
[26] J. H. He and X. H. Wu, Chaos, Solitons and Fractals **30**, 700 (2006); J. H. He and L. N. Zhang, Phys. Lett. A **38**, 1044 (2008); J. H. He and M. A. Abdou, Chaos, Solitons and Fractals **34**, 421 (2007); M. A. Abdou, A. A. Soliman, and S. T. EI-Basyony, Phys. Lett. A **369**, 469 (2007).
[27] M. Wadati and K. Sawada, J. Phys. Soc. Jpn. **48**, 312 (1980); M. Wadati and K. Sawada, J. Phys. Soc. Jpn. **48**, 319 (1980); K. Ohkuma and M. Wadati, J. Phys. Soc. Jpn. **52**, 749 (1983).
[28] Z. T. Fu, S. K. Liu, and S. D. Liu, Phys. Lett. A **20**, 72 (2001).
[29] E. G. Fan, Chaos, Solitons and Fractals **15**, 67 (2003).
[30] C. L. Bai, C. J. Bai, and H. Zhao, Z. Naturforsch. **60a**, 211 (2005); C. L. Bai, C. J. Bai, and H. Zhao, Commun. Theor. Phys. **44**, 821 (2005); C. L. Bai and H. Zhao, Commun. Theor. Phys. **41**, 521 (2004); H. Zhao and C. L. Bai, Chaos, Solitons and Fractals **30**, 217 (2006); H. Zhao, Chaos, Solitons and Fractals **36**, 1283 (2008).
[31] L. H. Zhang, L. H. Dong, and L. M. Yan, Appl. Math. Comput. **203**, 784 (2008).